THE CONTINUUM AS A TYPE OF ORDER: AN EXPOSITION OF THE MODERN THEORY*

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CHAPTER V

CONTINUOUS SERIES: ESPECIALLY THE TYPE OF THE REAL NUMBERS

54. In the preceding chapters we have considered the discrete series (§21) and the dense series (§41); we turn now to the study of the linear continuous series, which are the most important for algebra.

A *continuous series* in general is defined as any series (see §12 or §74) which satisfies Dedekind's postulate ($C1$, below) and the postulate of density ($C2$); a *linear* continuous series is then any continuous series which satisfies also a further condition, which I shall call the postulate of linearity ($C3$).

POSTULATE $C1$.† (*Dedekind's postulate.*) *If K_1 and K_2 are any two non-empty subclasses in K, such that every element of K belongs either to K_1 or to K_2 and every element of K_1 precedes every element of K_2, then there is an element X in K such that:*

1) *any element that precedes X belongs to K_1, and*

2) *any element that follows X belongs to K_2.*

This is the same as postulate $N1$ in §21.

POSTULATE $C2$. (*Postulate of density.*) *If a and b are elements of the class K, and $a < b$, then there is an element x such that $a < x$ and $x < b$.*

This is the same as postulate $H1$ in §41.

POSTULATE $C3$.‡ (*Postulate of linearity.*) *The class K contains a denumerable subclass R (§37) in such a way that between any two elements of the given class K there is an element of R.*

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* Presented to the American Mathematical Society at the Williamstown meeting, September 8, 1905.

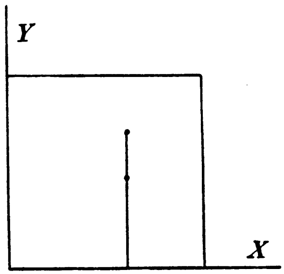
† R. Dedekind, *loc. cit.* (1872).

‡ G. Cantor, *loc. cit.* (1895), §11. O. Veblen replaces this postulate of linearity by two other postulates which he calls the pseudo-Archimedean postulate and the postulate of uniformity; see *Trans. Amer. Math. Soc.*, vol. 6 (1905), p. 165–171.

The consistency and independence of these postulates will be discussed in §63 and §64; postulate $C2$ is clearly redundant whenever postulate $C3$ is assumed.

55. The most familiar example of a linear continuous series is the class of points on a line, say one inch long, the relation $a < b$ signifying that a lies on the left of b. Dedekind's postulate is satisfied in this system, since if K_1 and K_2 are two subclasses of the kind described in the postulate, there will be a point of division on the line (either the last point of K_1 or the first point of K_2), which will serve as the point X demanded in the postulate. The postulate of density is also clearly satisfied, since between any two points of the line other points can be found. Finally, to see that the postulate of linearity holds, take as the subclass R the class of all points of the line whose distances from one end are rational fractions of an inch.

An example of a continuous series which is not linear is the class of all points (x, y) of a square (including the boundaries), arranged in order of magnitude of the x's, or, in case of equal x's, in order of magnitude of the y's. This series is continuous (satisfying postulates $C1$ and $C2$), but no subclass R of the kind demanded in postulate $C3$ is possible within it; for, if there were such a subclass it would have to contain elements corresponding to every point of the side of the square and therefore could not be denumerable (see §58 below).



Other examples, not depending on geometric intuition, will be given in §63 and §64, 3.

56. With the aid of the following definition, we may state two theorems that hold for all continuous series.

Definition. Let C be any non-empty subclass in any series $(K, <)$; if there is an element X in the series such that:

1) there is no element of C which follows X, while

2) if Y is any element preceding X there is at least one element of C which follows Y:—then this element X is called the *upper limit* of the subclass C.

If the subclass C happens to have a last element, this element itself will be the upper limit of the subclass. If C has no last element, it may or may

not have an upper limit; if it has an upper limit, then this upper limit is the element which comes *next after* the subclass C in the given series.*

THEOREM 1. *In any continuous series, if C is any subclass all of whose elements precede a given element, then C will have an upper limit in the series.*

Briefly, this theorem tells us that in any continuous series, every subclass which has *any* upper bound will have a *lowest* upper bound, — the terms "upper limit" and "lowest upper bound" being synonymous.

The full meaning of this theorem will be clearer after a study of the examples given in §§63–64 of series that are and those that are not continuous (compare also §50); the formal proof is easily given, as follows:

Under the conditions stated, the given series can be divided into two non-empty subclasses, K_1 and K^2, the first containing every element that is equaled or surpassed by any element of C, and the second containing all the other elements;† then by Dedekind's postulate there must be at least one element X "dividing" K_1 from K_2; moreover, there cannot be two such elements, for if there were, one would be the last element of K_1 and the other the first element of K_2, so that no element would lie between them (contrary to the postulate of density). This dividing element X is then the element required in the theorem.

Similarly, we may define the *lower limit* of a subclass, and prove the analogous theorem:

THEOREM 2. *In any continuous series, if C is any subclass all of whose elements follow a given element, then C will have a lower limit in the series.*

That is, in any continuous series, every subclass which has *any* lower bound will have a *highest* lower bound, or lower limit.

COROLLARY. *In any continuous series which has a first and a last element, every subclass will have both an upper and a lower limit in the series.*

57. The following theorem gives us another form of the definition of continuous series.

THEOREM.‡ *In the definition of a continuous series* (§54), *Dedekind's postulate may be replaced by the demand that every fundamental segment shall have a limit* (§49).

* It should be noticed that this definition of a limit of a subclass in general is quite consistent with the definition already given for the limit of a fundamental segment (§49).

† The subclass K_2 will not be an empty class, since by hypothesis there is at least one element in K which follows all the elements of C.

‡ *Cf.* a remark due to Whitehead in Russell's *Principles of Mathematics*, vol. 1 (1903), p. 299, footnote.

For, if the elements of the whole series are divided into two subclasses K_1 and K_2 as in the hypothesis of Dedekind's postulate, then K_1 (or K_1 without its last element, if it happens to have one) will be a fundamental segment, and the limit of this segment will correspond to the element X in Dedekind's postulate.

58. The next theorem concerns the infinitude of the elements of a continuous series.

THEOREM. *The elements of any continuous series* (§54) *form an infinite class which is not denumerable* (§37).

The proof, which is due to Cantor,* is as follows:

Suppose a given continuous series to be denumerable; then without disturbing the order of the elements we may attach to each one a definite natural number, using the notation $a(n)$ to represent the element corresponding to the number n.

We may assume without loss of generality that the element $a(1)$ precedes the element $a(2)$.

Then let p_1 and q_1 be the smallest numbers for which $a(p_1)$ and $a(q_1)$ lie between $a(1)$ and $a(2)$, and assume that the elements have been so numbered that $a(p_1) < a(q_1)$; then

$$a(1) < a(p_1) \quad < \quad a(q_1) < a(2).$$

Next, let p_2 and q_2 be the smallest numbers for which $a(p_2)$ and $a(q_2)$ lie between $a(p_1)$ and $a(q_1)$ and assume $a(p_2) < a(q_2)$, so that

$$a(1) < a(p_1) < a(p_2) \quad < \quad a(q_2) < a(q_1) < a(2).$$

And so on. In general, let p_{k+1} and q_{k+1} be the smallest numbers for which $a(p_{k+1})$ and $a(q_{k+1})$ lie between $a(p_k)$ and $a(q_k)$, and assume $a(p_{k+1}) < a(q_{k+1})$. In this way we determine a progression of elements $a(p_k)$ and a regression of elements $a(q_k)$, such that

$$a(1) < a(p_1) < a(p_2) < a(p_3) < \cdots \quad < \quad \cdots < a(q_3) < a(q_2) < a(q_1) < a(2).$$

Now since the series is continuous, the progression in question ought to have an upper limit (§56); but there is no element $a(n)$ which can serve as this upper limit, for if any element $a(n)$ is proposed, we can clearly carry the process just indicated so far that $a(n)$ will lie outside the interval $a(p_k) \cdots \quad \cdots a(q_k)$.

Therefore if the series is denumerable it cannot be continuous, and the theorem is proved.

59. The theorems of §§56–58 hold for all continuous series; the following theorems apply only to the linear continuous series.

THEOREM. *Every linear continuous series* (§54) *contains a subclass R*

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* G. Cantor, *Crelle's Journ. für Math.*, vol. 77 (1874), p. 260.

of type η (§44), *such that between any two elements of the given series there is an element of* R.

For, the denumerable subclass R whose existence is demanded in postulate $C3$, or the same subclass without its extreme elements if it has them, is clearly of type η (the type of the rational numbers).

This subclass R of type η may be called the *skeleton*, or *framework*, of the given series; the elements which belong to R may be called, for the moment, the *rational* elements, and those that do not belong to R the *irrational* elements of the series.

Since the class of all the elements of any continuous series is non-denumerably infinite (§58), it is clear that the rational elements of a linear continuous series cannot exhaust the series; in fact the class of irrational elements in any such series will itself be non-denumerably infinite (compare §38).

60. The most important property of the rational elements is given in the following theorem, which follows immediately from §56:

THEOREM. *In any linear continuous series, every element* a *(unless it be the first) determines a fundamental segment* (§46) *of the so-called rational elements, namely, the series of all the rationals preceding* a*; and conversely, every fundamental segment of rationals determines an element of the given series, namely, the upper limit of the segment* (§56).

The rational elements of the given series correspond to the fundamental segments which have limits in the series of rationals; the irrational elements correspond to the segments which have no limits in the series of rationals (§§49, 50). The denumerable dense series considered in the preceding chapter are not continuous, since, as we have seen in §50, they contain fundamental segments which have no limits; the theorem thus brings out clearly the sense in which the linear continuous series are "richer" in elements than the denumerable dense series.

61. The linear continuous series, like the discrete series or the denumerable dense series, can be divided into four groups, distinguished by the presence or absence of extreme elements; all the series of any one group are ordinally similar (see below), and therefore constitute a definite type of order. In particular, a *linear continuous series* (§54) *which has both a first and a last element* is called by Cantor a series of the *type* θ, or the type of the *linear continuum*.*

* G. Cantor, *loc. cit.* (1895), §11.

The proof that any two series of type θ are ordinally similar follows readily from the analogous theorem in regard to series of type η (§45).* For, by §59 each of the given series of type θ will contain a subclass of "rational" elements of type η; by §45 these subclasses of rationals can be brought into ordinal correspondence with each other; and by §60 every element (except the first) of each of the given series is uniquely determined as the limit of a fundamental segment of rationals.

It should be noticed, however, that this correspondence can be set up in an infinite number of ways, since not only the selection of rational elements from the given series, but also the correspondence between the two sets of rational elements, can be determined in an infinite number of ways.

62. Since the definition of the type θ here adopted differs in manner of approach, though not in substance, from the definition given by Cantor, I add, in this section, a statement of Cantor's definition in its original form.†

Every progression or regression which belongs to a given series is called by Cantor a *fundamental sequence* (*Fundamentalreihe*); any element which is the limit of any fundamental sequence (upper limit in the case of a progression, lower limit in the case of a regression), is called a *principal element* (*Hauptelement*) of the series.‡ If every fundamental sequence in a given series has a limit in the series, the series is said to be *closed* (*abgeschlossen*); if every element of the series is the limit of some fundamental sequence, the series is said to be *dense-in-itself* (*insichdicht*); and any series which is both *dense-in-itself* and closed is said to be *perfect* (*perfekt*). Finally, if a series is such that between any two elements there are other elements, the series is said to be *dense* (*überalldicht*).

The following theorems follow at once from these definitions:

1) If a series is closed, it will have a first and a last element, and will satisfy Dedekind's postulate (§54).

2) If a series satisfies Dedekind's postulate, and has both extreme elements, it will be closed.

3) If a series is dense, it will be also dense-in-itself.

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* G. Cantor, *loc. cit.* (1895), §11.

† G. Cantor, *loc. cit.* (1895), §§10–11. An earlier definition of the arithmetical continuum given by Cantor in *Math. Ann.*, vol. 5 (1872), p. 123 [*cf. ibid.*, vol. 21 (1883), pp. 572–576], involved extra-ordinal considerations, and need not concern us here.

‡ This definition of a fundamental sequence is inaccurately quoted by Veblen (*loc. cit.*, p. 171), who leaves out the regressions. Thus, in the series

$$2', 1' ; \ldots, -3, -2, -1, 0, +1, +2, +3, \ldots ; 1'', 2''$$

the element $1'$ would be a principal element according to Cantor's definition, but not according to Veblen's. [The same word, *Fundamentalreihe*, has been used by Cantor in another connection, in discussing irrational numbers; *Math. Ann.*, vol. 21 (1883), p. 567.]

On the other hand, the following facts should be noticed:

4) A series may satisfy Dedekind's postulate, and still not be closed, as witness the series of all integers, or the series of all real numbers.*

5) Further, a series may be perfect (that is, dense-in-itself and closed), and not be dense; as witness the series discussed in §52, 3 (with end-points), or the series of all real numbers from 0 to 3 inclusive with the omission of those between 1 and 2.

6) And again, a series may be dense-in-itself and dense, and not be closed, as for example the series of rational numbers (with or without extreme elements).

7) Finally, a series may be closed, and not be dense-in-itself, as for example any finite series, or a series like this:

$$2', 1'; \cdots, {}^{-}3, {}^{-}2, {}^{-}1, 0, {}^{+}1, {}^{+}2, {}^{+}3, \cdots; 1'', 2''.$$

By the aid of these definitions, Cantor then defines a *series of type θ* by the following two conditions:

A) the series must be perfect (that is, dense-in-itself, and closed); and

B) the series must contain a denumerable subclass R in such a way that between any two elements of the given series there is an element of R.†

The agreement between this definition and that given in §61 is then readily established, a closed dense series being the same as a continuous series with a first and a last element. The use of Dedekind's postulate instead of the postulate of closure implies the use of fundamental segments instead of the fundamental sequences; this modification of Cantor's method seems to me desirable, since every segment determines a unique element, and every element determines a unique segment, while in the case of the sequences, although every sequence determines a unique element, it is not true that every element determines a unique sequence.‡ I have preferred Dedekind's postulate to the postulate of §57 merely because of its greater symmetry.

Examples of linear continuous series.

63. The following examples serve to establish (compare §19) the consistency of the postulates of the present chapter (§54); in all but the first of them we avoid making any appeal to geometric intuition.

1) The simplest geometric example of a linear continuous series is the class of all points on a line, already considered in §55.

The most important non-geometrical examples are:

2) The class of (absolute) real numbers, arranged in the usual order; and

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* It is therefore perhaps unfortunate to speak of Dedekind's postulate as the postulate of closure.

† It will be noted that the first part of condition A is redundant, since every series which satisfies condition B will clearly be dense, and every dense series will be dense-in-itself.

‡ It can be shown, however, that the class of fundamental sequences in any continuous series has the same "cardinal number" §88 as the class of elements in the series itself (compare §71).

3) The class of *all* real numbers (positive, negative, and zero), arranged in the usual order.

By the *absolute real numbers* we mean the class of all fundamental segments (§46) in the series of absolute rational numbers (§51, 2); and by the usual order within this class we mean that a segment a shall precede a segment b when a is a part of b.*

This system clearly satisfies the general conditions for a series (§12), since if a and b are any two distinct fundamental segments of any dense series, one of them must be a part of the other, and the relation of part to whole is transitive. Further, the series is dense; for, if a segment a is part of a segment b, there will always be rationals belonging to b and not to a; a segment x containing the segment a and some of these rationals, will then lie "between" the segments a and b. To show that Dedekind's postulate is also satisfied, suppose that the whole series K is divided in any way into two subclasses K_1 and K_2 such that every element of K_1 precedes every element of K_2; then the class of all rationals which belong to any element of K_1 will be a fundamental segment in the series of rationals, and will be the element X demanded in the postulate. Finally, the series is a *linear* continuous series, since we may take as the required subclass R all the elements of K which have limits in the series of rationals (§49).

By the series of *all* real numbers (positive, negative, or zero) we then mean a series built up from the series of absolute real numbers in the same way that the series of all rationals was built up from the series of absolute rationals in

* This is the definition adopted by Russell (*loc. cit.*, chap. 33); it was first given in this form by M. Pasch (*Differential- und Integralrechnung*, 1882), his *Zahlenstrecke* (fundamental segment of rationals) being a modification of Dedekind's *Schnitt* or *cut* (1872). Similar definitions have been given by Dedekind (1872), Cantor (1872), Peano (1898), and others; see an historical account by Peano in *Rev. de Math.*, vol. 6 (1898), pp. 126–140. The construction of the system of (absolute) real numbers may be briefly described as follows (confining ourselves to the positive numbers): (1) the *integers* are the natural numbers, assumed as known; (2) the *rationals* are pairs of integers; and (3) the *reals* are classes (fundamental segments) of rationals. As a matter of convenience in notation, a pair of integers in which the denominator is 1 is represented by the numerator alone; rational numbers of this form are said to be *integral*, while all other rational numbers are called *fractional*. Again, a fundamental segment which has a limit in the series of rationals is represented by the same symbol as its limit; real numbers of this form are said to be *rational*, while all other real numbers are called *irrational* (compare §50). This notation, however, should not be interpreted as meaning that the class of real numbers *includes* the class of rationals, or that the class of rational numbers *includes* the class of integers. On the contrary, while the "integral number 2" means simply the second number in the natural series, the "rational number 2" means the pair of natural numbers 2 and 1, and "the real number 2" means the class of all rational numbers which precede the rational number 2/1. [The rules by which the sum and product of two real numbers are defined do not concern us here, in this discussion of the purely ordinal theory; see O. Stolz and J. A. Gmeiner, *Theoretische Arithmetik* (1901–), J. Tannery, *Introduction à la théorie des fonctions* (1886), or H. Weber and J. Wellstein, *Encyclopädie der Elementar-Mathematik* (vol. 1, 1903).]

§51, 3. Or again, all real numbers may be defined as fundamental segments of the series of all rationals, just as the absolute real numbers are defined as fundamental segments of the series of absolute rationals.

In the series of real numbers we have thus constructed an artificial system which certainly satisfies all the conditions for a linear continuous series (§54); there can therefore be no doubt that those conditions are free from inconsistency.* If we assume as geometrically evident that the series of all points on a line an inch long also satisfies these conditions, then an ordinal correspondence can be established between the real numbers and the points of the line, in accordance with §61 (taking as the "rational" points of the line those points whose distances from one end of the line are proper fractions of an inch); but in setting up this correspondence we must recognize that the continuity of the series of points on the line is an assumption which is not capable of direct experimental verification.

4) Another example of a linear continuous series is the class of all non-terminating decimal fractions, arranged in the usual order (§19, 9).

This series is dense; for, suppose a and b are any two of the decimals such that $a < b$; let β_k be the first digit of b which is greater than the corresponding digit of a, and let β_n be the first digit beyond β_k which is different from 0; then any decimal x in which the first $n-1$ digits are the same as in b, while the nth digit is less by one than β_n, will lie between a and b. Further, the series satisfies Dedekind's postulate; for, if K_1 and K_2 are the given subclasses, we may determine the decimal X $(= .\xi_1\xi_2\xi_3\cdots)$ as follows: ξ_1 is the largest digit which occurs in the first place of any decimal belonging to K_1; ξ_2 is the largest digit which occurs in the second place of any decimal beginning with ξ_1 and belonging to K_1; ξ_3 is the largest digit which occurs in the third place of any decimal beginning with $\xi_1\xi_2$ and belonging to K_1; and so on. Finally, the series is linear, since we may take as the subclass R the class of those decimals in which all the places after any given place are filled with 9's.—The series, as we notice, contains a last element ($.999\cdots$), but no first.

5) As a final example we mention the series described in §19, 8, namely: $K =$ the class of all possible infinite classes of the natural numbers, no number being repeated in any one class; with the relation $<$ so defined that $a < b$ when the smallest number in a is less than the smallest number in b, or when the smallest n numbers of a and b are the same and the $(n + 1)$st number or a is less than the $(n + 1)$st number of b.

*Cf. H. Weber, *Algebra*, vol. 1, p. 7, where the real numbers are defined (after Dedekind) as "cuts" in the series of rationals, instead of as fundamental segments of rationals. (A *cut* is simply a rule for dividing a series K into two subclasses K_1 and K_2, such that every element of K_1 precedes every element of K_2.)

This series is continuous, as the reader may readily verify; and it satisfies the postulate of linearity, since we may take as the subclass R the class of all the elements in which only a finite number of the natural numbers are absent. We notice also that the series contains a first element (namely the class of *all* the natural numbers), but no last element.

This example is particularly interesting as showing how a linear continuous series can be built up directly from the natural numbers, without making use of the rationals.*

Examples of series which are not linear continuous series.

64. The examples given in this section serve to show (compare §20) that postulates $C1$ and $C2$ (§54) are independent of each other, and that postulate $C3$ is independent of both of them. Postulate $C2$, on the other hand, is clearly a consequence of postulate $C3$.

1) *Dense series which do not satisfy Dedekind's postulate.*

a) Denumerable series which are dense but do not satisfy Dedekind's postulate are given in §51.

b) A non-denumerable example of the same sort is the series of all the points on a line with the exception of some single point; or better, the series described in §52, 2, *b*.

2) *Series which satisfy Dedekind's postulate, but are not dense.*

a) The series described in §52, 3 (consisting of the ternary fractions in which the digits 0 and 2 only are used) is not dense, but can readily be shown to satisfy the postulate of Dedekind.

b) Any discrete series is also an example of this kind.

3) *A continuous series which is not linear.*

Let K be the class of all couples (x, y), where x and y are real numbers from 0 to 1 inclusive; and let $(x_1, y_1) < (x_2, y_2)$ when $x_1 < x_2$, or when $x_1 = x_2$ and $y_1 < y_2$. This series is a continuous series (satisfying $C1$ and $C2$); but it is not a linear continuous series, since no denumerable subclass R of the kind demanded in postulate $C3$ is possible within it. (The same example, in geometric form, has been mentioned already in §55; other examples of the same kind will occur in Chapter VI.)

4) As a final example of a series which is not continuous, we mention a class K composed of two sets of real numbers, say red and blue, with a relation of order defined as follows: of two elements which have unequal numer-

* B. Russell, *Principles of Mathematics*, vol. 1, p. 299.

ical values, that one shall precede which would precede in the usual order of real numbers, regardless of color; of two elements which have the same numerical value, the red shall precede.

This system is built up by interpolating the elements of one continuous series between the elements of another continuous series; the resulting series, instead of being "more continuous" as one might have been tempted to expect, is no longer even dense, since every red element has an immediate successor (compare §52, 1, *b*).

Arithmetical operations among the elements of a continuous series.

65. In the case of continuous series as in the case of dense series it is not possible to give purely ordinal definitions of the sums and products of the elements; for, unless some other fundamental notion besides the notion of order is introduced, the elements of these series (except extreme elements) have *no intrinsic properties by which we can tell them apart* (compare §53). We might, to be sure, define sums and products of the elements of some particular series (like the series of real numbers, in the usual order) by the use of extra-ordinal properties peculiar to that series, and then transfer these definitions to other series of the same type by a one-to-one ordinal correspondence; but this method would be wholly inadequate, since the ordinal correspondence could be set up in an infinite number of ways. To construct a *completely determinate* continuous system it is therefore necessary to introduce some further notions, like addition and multiplication, besides the notion of order, as fundamental notions of the system.*

* See for example my set of postulates for ordinary complex algebra, *Trans. Amer. Math. Soc.*, vol. 6 (1905), pp. 209–229, especially §8.

CHAPTER VI

CONTINUOUS SERIES OF MORE THAN ONE DIMENSION, WITH A NOTE ON MULTIPLY ORDERED CLASSES

66. In the preceding chapters we have studied various kinds of series, or simply ordered classes (§12), — especially the linear continuous series (§54). In the following chapter we consider briefly some kinds of continuous series which are not linear, and add a short note on multiply ordered classes.

*Continuous series of more than one dimension.**

67. We shall use the term *one-dimensional framework* or *skeleton* (R_1) to denote a *series of type* η, that is, a denumerable dense series without extreme elements (§44). A *one-dimensional*, or *linear*, *continuous series* is then any continuous series which contains a framework R_1 in such a way that between any two elements of the given series there are elements of R_1 (§59).

Again, a *two-dimensional framework*, R_2, is any series formed from a one-dimensional continuous series by replacing each element of that series by a series of type η; and a *two-dimensional continuous series* is any continuous series which contains a framework R_2 in the same way.

And so on. In general, an *n-dimensional framework*, R_n, is any series formed from an $(n-1)$-dimensional continuous series by replacing each element of that series by a series of type η; and an *n-dimensional continuous series* is any continuous series which contains a framework R_n in such a way that between any two elements of the given series there are elements of R_n.

68. By a *k-dimensional section* of any continuous series we shall mean any segment (§47) which forms by itself a k-dimensional continuous series, but is not a part of any other such segment.†

In an n-dimensional continuous series each one-dimensional section, unless it be the $\genfrac{}{}{0pt}{}{\text{first}}{\text{last}}$, will have a $\genfrac{}{}{0pt}{}{\text{first}}{\text{last}}$ element, and these elements taken in order

* The study of the multi-dimensional continuous series was proposed by Cantor in *Math. Ann.*, vol. 21, p. 590, note 12 (1883), but seems never to have been carried out in detail. It would be interesting to extend the discussion to continuous series of a transfinite number of dimensions (*cf.* §88).

† We may speak of a section of a framework R_n, as well as of a section of a continuous series. A "zero-dimensional" section would be, of course, a single element. — If preferred, the word *constituent* may be used instead of *section*.

will form an $(n-1)$-dimensional continuous series. And so in general: each k-dimensional section, unless it be the $\substack{\text{first}\\ \text{last}}$, will have a $\substack{\text{first}\\ \text{last}}$ $(k-1)$-dimensional section, and these $(k-1)$-dimensional sections taken in order will be the elements of an $(n-k)$-dimensional continuous series.

69. As already noted, there are four different types of one-dimensional continuous series, distinguished by the presence or absence of extreme elements; in particular, a one-dimensional continuous series with both a first and a last element is called a series of *type* θ (§61).

A two-dimensional continuous series may or may not have a first one-dimensional section, and that section in turn may or may not have a first element. Similarly, there may or may not be a last one-dimensional section, which in turn may or may not have a last element. There are therefore nine different types of such series, distinguished by their initial and terminal properties. In particular, a two-dimensional continuous series with both a first and a last element I shall call a series of *type* θ^2 (since it may be formed from a series of type θ by replacing each element by another series of type θ).*

And so on. In general, there will be $(n+1)^2$ different types of n-dimensional continuous series, distinguished by their initial and terminal properties. In particular, an n-dimensional continuous series which has both a first and a last element may be called a series of *type* θ^n, or an *n-dimensional continuum.*

The proof that any two series of the same type are ordinally similar, and that all the types are really distinct, is readily obtained by an extension of the methods used in §§45 and 61.

70. An example of an n-dimensional continuous series is a class whose elements are sets of real numbers, $(x_1, x_2, x_3, \cdots, x_n)$, where x_1 is any real number, and $x_2, x_3, \cdots, x_n$ are restricted to the interval from 0 to 1 inclusive; the elements of the class being arranged primarily in order of the x_1's; or in case of equal x_1's, in order of the x_2's; or in case of equal x_1's and equal x_2's, in order of the x_3's; etc.

If $n = 1$, 2, or 3, the elements of this class can be represented geometrically: (1) by the points on a line; (2) by the points of a plane region bounded by two parallel lines; and (3) by the points of a space region bounded by a square prismatic surface. If n is greater than 3, no simple geometrical interpretation is possible.

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* *Cf.* Cantor's notation for the "product" of two normal series (§86).

71. Although the various types of series just considered are all distinct as types of order, yet it is important to notice that *the class of elements* of an n-dimensional continuous series can be put into one-to-one correspondence with *the class of elements* of a one-dimensional continuous series, if the relation of order is sacrificed; or, in the terminology of modern geometry, *the points of all space (of any number of dimensions) can be put into one-to-one correspondence with the points of a line.* One of Cantor's most interesting early discoveries was a device for actually setting up this correspondence; we give a sketch of the method for the case of two dimensions.*

As a preliminary step, we notice that a one-to-one correspondence can be set up between the points of any two lines, of length a and b, with or without end-points. For, each line can be divided into a denumerable set of segments of lengths equal, say, to $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$ of the length of the line; a one-to-one correspondence can be established between the two sets of segments, and then (as in §3) between the interior points of each segment of one set and the interior points of the corresponding segment of the other set; and a one-to-one correspondence can also be established between the two sets of points of division.

Consider now the points (x, y) within a square one inch on a side $(0 < x < 1, 0 < y < 1)$, and the points t on a line say three inches long $(0 < t < 3)$; and divide each third of the line t into a denumerable set of segments of lengths $\frac{1}{2}, \frac{1}{4}, \frac{1}{8}, \cdots$ of an inch. A one-to-one correspondence between the points of the square and the points of the line can then be established as follows:

1) The points (x, y) for which x and y are both rational form a denumerable set, and can therefore be put into one-to-one correspondence with the "rational" points of the line—that is, the points for which t is rational.

2) The points (x, y) for which x is rational and y irrational are the "irrational" points of a denumerable set of vertical lines, and can therefore be put into one-to-one correspondence with the "irrational" points of the denumerable set of segments which occupies, say, the last third of the line.

3) Similarly the points (x, y) for which y is rational and x irrational can be put into one-to-one correspondence with the "irrational" points of the middle third of the line.

4) Finally, the points for which x and y are both irrational can be put into one-to-one correspondence with the "irrational" points of the first third of the line. For, every *irrational* number a between 0 and 1 can be expressed as a *non-terminating* simple continued fraction, $a = [a_1, a_2, a_3, \cdots]$, that is:

$$a = \cfrac{1}{a_1 + \cfrac{1}{a_2 + \cfrac{1}{a_3 + \cdots,}}}$$

* Cantor, *Crelle's Journ. für Math.*, vol. 84, pp. 242–258 (1877); *cf. Math. Ann.*, vol. 46. p. 488 (1895).

where $x_1, x_2, x_3, \cdots$ are positive integers; so that to the point

$$x = [x_1, x_2, x_3, \cdots],$$
$$y = [y_1, y_2, y_3, \cdots]$$

in the square we can assign the point

$$t = [x_1, y_1, x_2, y_2, x_3, y_3, \cdots]$$

on the line; while inversely, to the point

$$t = [t_1, t_2, t_3, \cdots]$$

on the line we assign the point

$$x = [t_1, t_3, t_5, \cdots],$$
$$y = [t_2, t_4, t_6, \cdots]$$

in the square.

Thus the correspondence between the points of the square and the points of the line is complete; and the method is easily extended to any number of dimensions, finite or denumerably infinite.

*Note on multiply ordered classes.**

72. A *multiply ordered class* is a system (§11) consisting of a class K the elements of which may be ordered according to several different serial relations.

For example, a class of musical tones may be arranged in order according to pitch, or according to intensity, or according to duration. Again, the class of points in space may be ordered according to their distances from three fixed planes.

A multiply ordered class may also be called a *multiple series;* but a system of this kind is not strictly a series with respect to any one of its ordering relations, since postulate 1 does not strictly hold (see §12 or §74).

The only discussion of multiply ordered classes which I know of is contained in Cantor's memoir of 1888.

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* This subject is studied in detail by Cantor in *Zeitschr. f. Phil. u. philos. Kritik*, vol. 92, pp. 240–265 (1888).

APPENDIX

NORMAL SERIES AND CANTOR'S TRANSFINITE NUMBERS

73. In §§21, 41, and 54, certain special kinds of series ("discrete," "dense," "continuous") have been defined, and their chief properties discussed.

In this appendix a brief account is now to be given of another special kind of series, which has proved to be of fundamental importance in Cantor's theory of the transfinite numbers, and I hope that some readers may be led, by this brief introduction, to a further study of that most recent development of mathematical thought, in which many problems of fundamental interest still await solution.

Th theory of the transfinite numbers was created by Georg Cantor in 1883, in a monograph called *Grundlagen einer allgemeinen Mannichfaltigkeitslehre; ein mathematisch-philosophischer Versuch in der Lehre des Unendlichen.* A much clearer presentation of the subject will be found in his *Beiträge zur Begründung der transfiniten Mengenlehre* in the *Mathematische Annalen* (1895, 1897); but many of the speculations which were begun or suggested in the *Grundlagen* have not yet been developed.*

* The earliest of Cantor's writings which bear upon this subject will be found in *Math. Ann.*, vol. 5, pp. 123–132 (1872); and in Crelle's (or Borchardt's) *Journ. für Math.*, vol. 77, pp. 258–262 (1874); vol. 84, pp. 242–258 (1877). Then came a series of six articles "Über unendliche, lineare Punktmannichfaltigkeiten," *Math. Ann.*, vol. 15, pp. 1–7 (1879); vol. 17, pp. 355–358 (1880); vol. 20, pp. 113–121 (1882); vol. 21, pp. 51–58 (1883); vol. 21, pp. 545–591 (1883); vol. 23, pp. 453–488 (1884). The fifth of these articles is identical with the monograph published in the same year (1883) under the title "Grundlagen einer allgemeinen Mannichfaltigkeitslehre"—page n of the "Grundlagen" corresponding to page $(n + 544)$ of the article in the *Annalen.* [All the articles mentioned thus far, or partial extracts from them, are translated into French in the *Acta Mathematica*, vol. 2, 1883. The same journal contains also some further contributions; see vol. 2, pp. 409–414 (1883); vol. 4, pp. 381–392 (1884); vol. 7, pp. 105–124 (1885).] These articles were followed by a number of writings in defence of the new theory; see especially the *Zeitschrift für Phil. und philos. Kritik*, vol. 88, pp. 224–233 (1886); vol. 91, pp. 81–125, 252–270 (1887); vol. 92, pp. 240–265 (1888). Then came a short but interesting note in the *Jahresber. d. D. Math.-Ver.*, vol. 1, pp. 75–78 (1892), and finally the "Beiträge," etc., *Math. Ann.*, vol. 46, pp. 481–512 (1895); vol. 49, pp. 207–246 (1897); French translation by F. Marotte (1899). Since 1897 the literature of the subject has rapidly increased (Bernstein, Schröder, Burali-Forti, Borel, Schönflies, Hardy, Young, Russell, Whitehead, Jourdain, Zermelo, König, Hobson, and Veblen being among the contributors), but nothing further has been published by Cantor himself.—Special reference should be made to Schönflies's "Bericht über d. Entwickelung d. Lehre von d. Punktmannigfaltigkeiten," in the *Jahresber. d. D. Math.-Ver.*, vol. 8, part 2, 1900 (pages 1–250); and to Russell's *Principles of Mathematics*, vol. 1 (1903).

74. A *series*, or *simply ordered class*, has been defined in §12 as any system (K, $<$) which satisfies the following three conditions:

POSTULATE 1. *If a and b are distinct elements of the class K, then either $a < b$ or $b < a$.*

POSTULATE 2. *If $a < b$, then a and b are distinct.*

POSTULATE 3. *If $a < b$ and $b < c$, then $a < c$.*

A *normal series*, or *normally ordered class* (*wohlgeordnete Menge*, "*well-ordered*" *class* or *collection*), is then any series which satisfies the following three conditions:*

POSTULATE 4. *The series has a first element* (§17).

POSTULATE 5. *Every element, unless it be the last, has an immediate successor* (§17).

POSTULATE 6. *Every fundamental segment of the series has a limit.*†

The consistency and independence of these postulates are established by the examples already given in §§28–29.

In a normal series, any element which is the limit of a fundamental segment (and therefore has no immediate predecessor) is called a *limiting element* of the series (*Grenzelement*, *Element der zweiten Art*‡). Every element which is neither a limiting element, nor the first element of the series, will have a predecessor.

For example, the series

$$1_1,\ 2_1,\ 3_1,\ \cdots;\quad 1_2,\ 2_2,\ 3_2,\ \cdots;\quad 1_3,\ 2_3,\ 3_3,\ \cdots;\quad \cdots;\ 1'$$

is a normal series in which the limiting elements (1_2, 1_3, $\cdots$; $1'$) form a progression followed by a last element $1'$.

75. From postulates 1–6 it follows at once that Dedekind's postulate (see §21 or §54) will hold true in any normal series; indeed *we may use*

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* G. Cantor, *Math. Ann.*, vol. 21 (1883), p. 548; *ibid.*, vol. 49 (1897), p. 207. The name "normal series" was suggested to me by the adjective "normally ordered," which is used by E. W. Hobson as a translation of *wohlgeordnet;* see *Proc. Lond. Math. Soc.*, ser. 2, vol. 3 (1905), p. 170.

† A "fundamental segment" is any lower segment which has no last element; the "limit" of a fundamental segment is the element next following all the elements of the segment (§46, 49).

‡ G. Cantor, *Math. Ann.*, vol. 49, p. 226 (1897). Jourdain uses *Limes; Phil. Mag.*, ser. 6, vol. 7, p. 296 (1904). Compare §62, above.

Dedekind's postulate in place of postulate 6 *in the definition of a normal series* ;* I prefer postulate 6 in this case, however, because it emphasizes the unsymmetrical character of the normal series.

76. Other, very convenient, forms of the definition are the following:

1) *A normal series is any series in which every subclass* (§6) *has a first element.*†

2) *A normal series is any series which contains no subclass of the type* $*\omega$ (§25).‡

The equivalence of each of these definitions with the definition in §74 is easily verified.

Examples of normal series.

77. The simplest examples of normal series are those which contain only a finite number of elements; and since two finite series are ordinally similar when and only when they have the same number of elements, there will be a distinct type of normal series corresponding to every natural number (compare §27).

The simplest example of a normal series with an infinite number of elements is a series of type ω, that is, a progression (§24).

78. Other examples of normal series are the following:

A progression of series each of which is itself of type ω forms a series of type ω^2:

$$1,\ 2,\ 3,\ \cdots \mid 1,\ 2,\ 3,\ \cdots \mid 1,\ 2,\ 3,\ \cdots \mid \cdots.$$

A progression of series each of which is of type ω^2 forms a series of type ω^3:

$$1, 2, \cdots \mid 1, 2, \cdots \mid \cdots \| 1, 2, \cdots \mid 1, 2, \cdots \mid \cdots \| 1, 2, \cdots \mid 1, 2, \cdots \mid \cdots \| \cdots.$$

So in general; a progression of series each of which is of type ω^ν forms a series of type $\omega^{\nu+1}$, where ν is any positive integer.

Any type ω^ν can be represented by a series of points on a line of length a by the following device, illustrated for the case of type ω^3. First, divide the given line into a denumerable set of intervals, as most conveniently by

* O. Veblen, *Trans. Amer. Math. Soc.*, vol. 6, p. 170 (1905).

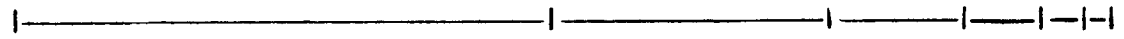
† Cantor, *loc. cit.* (1897), p. 208.

‡ Ph. Jourdain, *Phil. Mag.*, ser. 6, vol. 7, p. 65 (1904).

the set of points whose distances from the right-hand end of the line are

$$\frac{a}{2},\ \frac{a}{4},\ \frac{a}{8},\ \frac{a}{16},\ \cdots;$$

the points of division will form a series of type ω. Next, divide each interval into a denumerable set of intervals in a similar way; all the points of divi-



sion taken together will form a series of type ω^2. Finally, repeating the same operation once again, we obtain a series of points of type ω^3.

79. A series of the type called ω^ω may now be constructed as follows: Take a line of length a, and divide it into a denumerable set of intervals as above; in the first of these intervals insert a series of type ω, in the second a series of type ω^2, in the third a series of type ω^3, and so on; the total collection of points thus determined forms a series of type ω^ω.

A series of type ω^ω each of whose elements is a series of type ω^ω forms a series of type $(\omega^\omega)^2$ or $\omega^{\omega\cdot 2}$.

A series of type ω^ω each of whose elements is a series of type $\omega^{\omega\cdot 2}$ forms a series of type $\omega^{\omega\cdot 3}$.

And so in general a series of type ω^ω each of whose elements is a series of type $\omega^{\omega\cdot\nu}$ forms a series of type $\omega^{\omega(\nu+1)}$.

A series of the type called ω^{ω^2} can now be constructed as follows: Divide a given line into a denumerable set of intervals as before; in the first of these intervals insert a series of type ω^ω, in the second a series of type $\omega^{\omega\cdot 2}$, in the third a series of type $\omega^{\omega\cdot 3}$, and so on; the total collection of points thus determined forms a series of type $\omega^{\omega\cdot\omega}$ or ω^{ω^2}.

A series of type ω^{ω^2} each of whose elements is a series of type ω^{ω^2} forms a series of type $(\omega^{\omega^2})^2$ or $\omega^{\omega^2\cdot 2}$.

A series of type ω^{ω^2} each of whose elements is a series of type $\omega^{\omega^2\cdot 2}$ forms a series of type $\omega^{\omega^2\cdot 3}$.

And so in general a series of type $\omega^{\omega^2\cdot\nu}$ may be constructed, and hence a series of the type $\omega^{\omega^2\cdot\omega}$ or ω^{ω^3}, by another application of the denumerable set of intervals.

By an extension of the same methods we can thus construct series of each of the types originally denoted by $\omega_1, \omega_2, \omega_3, \cdots$, where

$$\omega_1 = \omega, \quad \omega_2 = \omega^{\omega_1}, \quad \omega_3 = \omega^{\omega_2}, \quad \cdots.^*$$

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* Cantor, *loc. cit.* (1897), p. 242. It should be noted that this notation has recently been abandoned, the subscripts under the ω's being now used for another purpose; see §83.

And so on *ad infinitum;* but none of the normal series thus constructed will contain more than a denumerable infinity of elements (compare §38).

80. In order to understand one further matter of notation, consider a normal series of the type represented, say, by

$$\omega^3 \cdot 5 + \omega^2 \cdot 7 + \omega + 2.$$

Here the plus signs indicate that the series is made up of four parts, in order from left to right; the first part consists of a series of type ω^3 taken five times in succession; the second part consists of a series of type ω^2 taken seven times in succession; the third part is a single series of type ω; and the last part is a finite series containing two elements. — And so in general the notation

$$\omega^{\mu} \cdot \nu_0 + \omega^{\mu - 1} \cdot \nu_1 + \omega^{\mu - 2} \cdot \nu_2 + \cdots + \nu_{\mu},$$

where μ is a positive integer, and the coefficients $\nu_0, \nu_1, \nu_2, \cdots, \nu_{\mu}$ are positive integers or zero, is to be interpreted in a similar way.*

It will be noticed that in the case of a progression, or of any normal series of the types described in §§78–79, the whole series is ordinally similar to each of its upper segments (§47); that is, if we cut off any lower segment from the series, the type is not altered. This will not be true in the case of the normal series of the types described in the present section.

General properties of normal series.

81. The fundamental properties of normal series are developed very carefully and clearly in Cantor's memoir of 1897; I mention the following theorems as perhaps the most important:

1) Every subclass in a normal series is itself a normal series.

2) If each element of a normal series is replaced by a normal series, and the whole regarded as a single series, the result will be still a normal series (compare the examples in §§78–79).

(These two theorems follow at once from the definition in §76, 1.)

Definition. The part of a normal series preceding any given element a is called a *lower segment* (*Abschnitt*) of the series (compare §47).†

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* Cantor, *loc. cit.* (1897), p. 229.

† Most writers, including Russell, translate *Abschnitt* by *segment* (without qualifying adjective); but since the word "segment" is already used in several different senses (see, for example, Veblen, *loc. cit.*, p. 166), it has seemed to me safer to use the longer term "lower segment," about which there can be no ambiguity.

3) A normal series is never ordinally similar to any one of its lower segments, or to any part of any one of its lower segments.

4) If two normal series are ordinally similar, the ordinal correspondence between them can be set up in only one way (compare §§26, 45, 61, and §§53, 65).

5) Any subclass of a normal series is ordinally similar to the whole series or else to some one of its lower segments.

6) If any two normal series, F and G, are given, then either F is ordinally similar to G, or F is ordinally similar to some definite lower segment of G, or G is ordinally similar to some definite lower segment of F; and these three relations are mutually exclusive. In the first case, F and G are of the same type; in the second case, F is said to be *less than* G; and in the third case, G is said to be *less than* F.

82. By virtue of this theorem 6, *the various types of normal series, when arranged "in the order of magnitude" (as defined in the theorem), form a series* (§74) *with respect to the relation "less than;" and, as Cantor has shown, this series will be itself a normal series.*

Moreover, by theorem 1, every possible collection of types of normal series, arranged in order of magnitude, will be itself a normal series.

Classification of the normal series.

83. The following classification of the normal series is a characteristic feature of Cantor's theory; since, however, the method of procedure, when pushed to its logical extreme, leads to apparently insurmountable difficulties, the whole scheme is regarded with a certain measure of suspicion.*

In the first place, every normal series in which the number of elements is finite is said to belong to the FIRST CLASS *of normal series.*

Now take all the types of series belonging to the first class, and arrange them in order of magnitude (§82); the result is a normal series of a certain type, called ω (compare §24).

Then *every normal series whose elements can be put into one-to-one correspondence* (§3) *with the elements of* ω *is said to belong to the* SECOND CLASS. In particular, the series of type ω are the *smallest* series of the second class.

* See E. Borel, *Leçons sur la théorie des fonctions* (1898), pp. 119–122, and a remark in Liouville's *Journ. de Math.*, ser. 5, vol. 9 (1903), p. 330; D. Hilbert, *Jahresb. d. D. Math.-Ver.*, vol. 8, p. 184 (1899); and especially E. W. Hobson, *Proc. Lond. Math. Soc.*, ser. 2, vol. 3, pp. 170–188 (June, 1905).

Next, take all the types of series belonging to the second class, and arrange them in order of magnitude; the resulting series is a normal series of a certain type, called ω_1 (or Ω).

Then *every normal series whose elements can be put into one-to-one correspondence with the elements of* ω_1 *is said to belong to the* THIRD CLASS. In particular, the series of type ω_1 are the *smallest* series of the third class.

And so on. In general, *every normal series whose elements can be put into one-to-one correspondence with the elements of* ω_ν (where ν is any positive integer) *is said to belong to the* $(\nu + 2)$th CLASS; and the series of type ω_ν will be the *smallest* series of that class.*

Moreover, by an extension of the device already employed several times, we can define a class of normal series whose smallest type would be denoted by ω_ω, or even ω_{ω_1}; and so on, *ad infinitum;* so that when we speak of the nth class of normal series, n need not be a positive integer, but may itself denote the type of any normal series.

84. In order to justify this classification, it is necessary to show that the classes described are really all distinct, so that no normal series belongs to more than one class; and further, that normal series belonging to each class actually exist, so that no class is "empty." Cantor has completed this investigation only as far as the first and second classes; problems concerning the existence of the higher classes, and the question whether every collection can be arranged as a normal series, are being actively debated at the present time (see, for example, articles by König, Schönflies, Bernstein, Borel, and Jourdain, in the current volume of the *Mathematische Annalen*,† and especially the article by Hobson already cited).

Each of the examples mentioned above is a normal series of the first or second class (since the number of elements in each case is at most denumerable, in view of §38); no similar example of a series of even the third class has yet been satisfactorily constructed.‡

85. The various classes of normal series can also be defined by purely ordinal postulates, as Veblen has shown how to do in his recent memoir.§

* The notation ω_ν for the smallest type of the $(\nu + 2)$th class was introduced by Russell, *Principles of Mathematics*, vol. 1 (1903), p. 322; compare Jourdain, *loc. cit.*, p. 295. The symbols ω and Ω were first used in this connection by Cantor in *Math. Ann.*, vol. 21, pp. 577, 582 (1883).

† *Math. Ann.*, vol. 60 (1905), pp. 177, 181, 187, 194, 465.

‡ See Hobson, *loc. cit.*, pp. 185–187.

§ O. Veblen, *Trans. Amer. Math. Soc.*, vol. 6, p. 170 (1905).

Thus, a normal series of the *first class* is any normal series which satisfies not only the postulates 1–6 of §74, but also the further conditions 7_1 and 8_1; namely:

POSTULATE 7_1. *Every element except the first has a predecessor* (§17).

POSTULATE 8_1. *There is a last element* (§17).

The *type* ω is then defined by postulates 1–6 with 7_1 and $8_1'$, where $8_1'$ is the contradictory of 8_1.

Next, a normal series of the *second class* is any normal series which satisfies 7_2 and 8_2, namely:

POSTULATE 7_2. *Every element except the first either has a predecessor or is the upper limit of some subclass of type* ω *(as just defined).*

POSTULATE 8_2. *There is either a last element, or a subclass of type* ω *which surpasses any given element of the series.**

The *type* ω_1 is then defined by postulates 1–6 with 7_2 and $8_2'$, where $8_2'$ is the contradictory of 8_2.

Similarly, a normal series of the *third class* is any normal series which satisfies 7_3 and 8_3, namely:

POSTULATE 7_3. *Every element except the first either has a predecessor, or is the upper limit of some subclass of type* ω, *or is the upper limit of some subclass of type* ω_1.

POSTULATE 8_3. *There is either a last element, or a subclass of type* ω *which surpasses any given element, or a subclass of type* ω_1 *which surpasses any given element.*

The *type* ω_2 is then defined by postulates 1–6 with 7_3 and $8_3'$, where, as before, $8_3'$ denotes the contradictory of 8_3.†

And so on. The establishment of complete sets of postulates like these seems to me an essential step toward the solution of the difficult problems connected with this subject.

The transfinite ordinal numbers.

86. It is now easy to explain what is meant by the *ordinal numbers* (*Ordnungszahlen*), in the generalized sense in which Cantor now uses that term: *they are simply the various types of order exhibited by the normal series.*‡ In other words, according to the theory of Russell, the ordinal number corresponding to any given normal series is the *class of all series which are ordi-*

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* That is, if x is any element of the given series, there is an element y in the subclass for which $x < y$.

† Thus, postulate $8_3'$ contains three separate statements: first, there is no last element; secondly, every subclass of type ω has an upper limit; and thirdly, every subclass of type ω_1 has an upper limit.

‡ *Loc. cit.* (1887), p. 84; and *loc. cit.* (1897), p. 216.

nally similar to the given series; any one of these ordinally similar series may be taken to represent the ordinal number of the given series.*

The ordinal numbers of the *first class* (§83) are the *finite* ordinal numbers, with which we have always been familiar; the ordinal numbers of the *second or higher classes* are the *transfinite* ordinal numbers created by Cantor, and which constitute, in a certain true sense, "*eine Fortsetzung der realen ganzen Zahlenreihe über das Unendliche hinaus.*" † The smallest of the transfinite ordinals is ω.

By the *sum*, $a + b$, of two ordinal numbers, a and b, is meant simply the type of series obtained when a series of type a is followed by a series of type b and the whole regarded as a single series.‡ Clearly $a + b$ will not always be the same as $b + a$ (for example, $1 + \omega = \omega$, while $\omega + 1$ is a new type); but always $(a + b) + c = a + (b + c)$.

By the *product*, ab, of an ordinal number a multiplied by an ordinal number b, is meant the type of series obtained as follows: in a series of type b replace each element by a series of type a, and regard the whole as a single series; the result will be a normal series (by §81, 2), and the type of this normal series is what is meant by ab.§ Clearly ab will not always equal ba (for example, $2\omega = \omega$, while $\omega \cdot 2$ is a new type); but always $(ab)c = a(bc)$, and also $a(b + c) = ab + ac$, although not $(b + c)a = ba + ca$.

The definition of a^b, where a and b are general ordinal numbers is too complicated to repeat in this place.|| Enough has at any rate been said to give at least some notion of the nature of the artificial algebra which Cantor has here so ingeniously constructed.

The transfinite cardinal numbers.

87. For the sake of completeness I add here a brief note on the meaning of some of the terms in Cantor's theory of the (generalized) cardinal numbers.¶ This theory has nothing to do with series, or *ordered* classes, but is a development of the theory of *classes as such* (§11); nevertheless the difficulties met with in this theory are closely analogous to the difficulties we have pointed out in the theory of the ordinal numbers (§84), and it is impossible to read the literature of either theory without some acquaintance with the other.

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* Russell, *Principles of Mathematics*, vol. 1 (1903), p. 312.

† *Loc. cit.* (1883), p. 545. On an extension of the term *abzählbar*, see *Math. Ann.*, vol. 23, p. 456.

‡ *Loc. cit.* (1883), p. 550.

§ In Cantor's earlier definition of the product ab, a was the multiplier (*loc. cit.*, 1883, p. 551); the order was changed in his later articles, so that a is now the multiplicand (see *loc. cit.*, 1887, p. 96, and *loc. cit.*, 1897, pp. 217, 231).

|| See *Math. Ann.*, vol. 49 (1897), p. 231.

¶ The standard account of this theory is in Cantor's article of 1895.

88. If two classes can be brought into one-to-one correspondence (§3), they are said to be *equivalent* (*äquivalent*). For example, the class of rational numbers is equivalent to the class of positive integers (compare §19, 6); or the class of points on a line is equivalent to the class of all points in space (§71).

The *cardnial number* (*Mächtigkeit*) of a given class A is then defined as *the class of all those classes which are equivalent to* A.* The *finite* cardinal numbers are the cardinal numbers which belong to finite classes; the *transfinite* cardinals are those which belong to infinite classes (§7).

According to this definition, if two classes A and B are *equivalent*, their cardinal numbers will clearly be *identical*.

If a class A is equivalent to a part of a class B, but not to the whole, then A is said to be *less than* B; in this case the cardinal number of A will be *less than* the cardinal number of B.

We cannot, however, affirm that all cardinal numbers can be arranged as a series, in order of magnitude, for while postulates 2 and 3 (§74) clearly hold with regard to the relation "less than" as just defined, postulate 1 has never been proved. In other words, non-equivalent classes may possibly exist, neither of which is "less than" the other.†

On the other hand, Cantor has proved that when any class is given, a class can be constructed which shall have a greater cardinal number than the given class.‡

For example, let C denote the class of elements in a linear continuum, say the class of points on a line one inch long (compare §71); and let C' denote the class of all possible "bi-colored rods" which can be constructed by painting each point of the given line either red or blue. Then the class of rods, C', has a higher cardinal number than the class of points, C, as may be proved as follows:

In the first place, C *is equivalent to a part of* C'; for example, to the class of rods in which one point is painted red and all the other points blue. Secondly, C *is not equivalent to the whole of* C'; for, if any alleged one-to-one correspondence between the rods and the points were proposed, we could at once define a rod which would not be included in the scheme: namely, the rod in which the color of each

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* The term *Mächtigkeit* was first used by Cantor in *Crelle*, vol. 84, p. 242 (1877). Power, potency, multitude, and dignity are some of the English equivalents. The term *Cardinalzahl* was introduced in 1887. The notion of a cardinal number as a *class* is emphasized by Russell.

† Compare E. Borel, *Lecons sur la théorie des fonctions* (1898), pp. 102–110.

‡ Cantor, *J. d. D. Math.-Ver.*, vol. 1 (1892), p. 77; E. Borel, *loc. cit.*, (1898), p. 107.

point x is opposite to the color of the point x in the rod which is assigned to the point x of the given line; this rod would differ from each rod of the proposed scheme in the color of at least one point.

The class C' has therefore a higher cardinal number than the class C. It is not known, however, whether there may not be other classes whose cardinal numbers lie between the cardinal numbers of C and C'.

89. Of special interest are the cardinal numbers of the various types of normal series; but when we speak of the cardinal number of a *series*, it must be understood that we mean the cardinal number of the *class of elements which occur in the series*, without regard to their order.

The cardinal numbers of the finite normal series are the *finite cardinal numbers*, with which we have always been familiar.

The cardinal number of a series of type ω is denoted by the Hebrew letter Aleph with a subscript 0:*

$$\aleph_0.$$

This $\aleph_0$ will then be the cardinal number of any normal series of the second class, since all the series of the second class are, by definition, equivalent.

The cardinal number of a series of type ω_1 (or Ω) is denoted by $\aleph_1$; this will then be the cardinal number of any normal series of the third class.

And so on. In general, the cardinal number of a series of type ω_ν is denoted by $\aleph_\nu$; this will then be the cardinal number of any normal series of the $(\nu + 2)$th class.

If we assume the series of classes of ordinal numbers (§86), we thus obtain a series of cardinal numbers

$$\aleph_0, \aleph_1, \cdots, \aleph_\omega, \cdots,$$

arranged in order of increasing magnitude; this series will be a normal series with respect to the relation "less than," and ordinally similar to the series of ordinal numbers; but all the difficulties that are involved in the one series are involved in the other. In particular, it requires proof to show that two Alephs, as $\aleph_\nu$ and $\aleph_{\nu+1}$, are really non-equivalent, and that no other cardinal number lies between them. Cantor has shown merely that $\aleph_0$ is the *smallest* transfinite cardinal number, and that $\aleph_1$ is the number *next greater*.† Again, the vexed question: *can the cardinal number of the linear continuum* (§54) *be found among the Alephs?* is equivalent to the question: *can the class of ele-*

* Cantor, *loc. cit.* (1895), p. 492.

† *Math. Ann.*, vol. 21, p. 581 (1883).

*ments in the continuum be arranged in the form of a normal series?** All that is known at present on this point is that the cardinal number of the continuum is *not less* than $\aleph_1$.

90. We speak next of the sums and products of the cardinal numbers.†

The *sum* $A + B$ of two classes A and B which have no common element is the class containing all the elements of A and B together.

If a and b are the cardinal numbers of two such classes A and B, the *sum*, $a + b$, of these two cardinals is then defined as the cardinal number of $A + B$. Clearly $a + b = b + a$, and $(a + b) + c = a + (b + c)$.

The *product*, AB, of two classes A and B which have no common element is the class of all couples (α, β), where α is any element of A, and β any element of B.

If a and b are the cardinal numbers of two such classes, the *product*, ab, of these two cardinals is then defined as the cardinal number of AB. Clearly, $ab = ba$, $(ab)c = a(bc)$, and $a(b + c) = ab + ac$.

Finally, A^B denotes the class of all *coverings* (*Belegungen*) of B by A, where a "covering" of B by A is any law according to which each element of B determines uniquely an element of A (not excluding the cases in which various elements of B may determine the same element of A).‡

The b^{th} *power of* a, a^b, where a and b are the cardinal numbers of any two classes A and B, is then defined as the cardinal number of A^B. Clearly $a^b a^c = a^{b+c}$, $(a^b)^c = a^{bc}$, and $(ab)^c = a^c b^c$.

In this way Cantor has constructed an artificial algebra of the cardinal numbers, analogous to the algebra of the ordinal numbers, but resembling much more closely the familiar algebra of the finite integers.

Perhaps the most famous result obtained in this algebra is the formula §

$$c = 2^{\aleph_0},$$

where c stands for the cardinal number of the continuum, and $2^{\aleph_0}$ is determined according to the rule just stated for the powers of cardinal numbers. It becomes an important question, therefore, to decide whether

$$2^{\aleph_0} = \aleph_1$$

or not (compare §89, end).

91. In conclusion, it may be well to repeat that when we speak of a *cardinal* number, we always mean the cardinal number *of some given class;* and when we speak of an *ordinal* number, we always mean the ordinal number *of some given normal series.*

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* The question whether every class can be normally ordered was first proposed by Cantor in *Math. Ann.*, vol. 21. p 550 (1883); the proof given by Zermelo in *Math. Ann.*, vol. 59, p. 514 (1904), has not been generally accepted. For the most recent discussion, see the papers cited in §84.

† *Zeitschr. f. Phil. u. philos. Kritik*, vol. 91, pp. 120–121 (1887); *Math. Ann.*, vol. 46, p. 485 (1895).

‡ *Math. Ann.*, vol. 46, p. 487 (1895).

§ *Math. Ann.*, vol. 46, p. 488 (1895).

Whether these new concepts will find important applications in practical problems is a question for the future to decide. (The *elementary* parts of Cantor's work have already proved useful, indeed almost indispensable, in the theory of functions of a real variable.)

INDEX OF TECHNICAL TERMS

The numerals refer to the sections.

A bibliography of Cantor's writings on these subjects will be found under §73.

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